Sections 13.1–13.2

Vector-Valued Functions

THE UNIVERSITY OF KANSAS
by Joseph Phillip Brennan
Jila Niknejad

1 Scalar and Vector Functions

Vector-Valued Functions

A **vector function** (or **vector-valued function**) is a function whose output is a vector.

For example, a vector function $\vec{r} : \mathbb{R} \to \mathbb{R}^3$ has the form

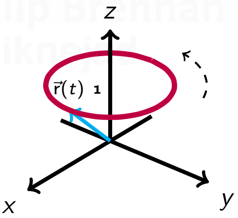
$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$$

where the independent variable t is a scalar in $\mathbb{R}$ and the dependent variable $\vec{r}(t)$ is a vector in $\mathbb{R}^3$. The scalar functions f, g, and h are the **components** of the vector function $\vec{r}$.

Example 1: The vector function

$$\vec{r}(t) = \langle \cos(t), \sin(t), 1 \rangle$$

has domain $\mathbb{R} = (-\infty, \infty)$. Its range is a circle of radius 1 in the plane $z = 1$.



Limits and Continuity for Vector Functions

Let $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ be a function $\mathbb{R} \to \mathbb{R}^3$. The **limit** of $\vec{r}(t)$ as $t \to a$ is defined component-wise:

$$\lim_{t \to a} \vec{r}(t) = \left\langle \lim_{t \to a} f(t),\ \lim_{t \to a} g(t),\ \lim_{t \to a} h(t) \right\rangle$$

The function $\vec{r}$ is **continuous** at a if f, g, and h are all continuous at a.

Example 2: The components of the vector function

$$\vec{r}(t) = \langle f(t),\ g(t),\ h(t) \rangle = \left\langle t,\ \ln(1+t),\ \frac{1}{t-1} \right\rangle$$

are all elementary functions, so they are continuous on their domains, namely

$$(-\infty, \infty), \qquad (-1, \infty), \qquad (-\infty, 1) \cup (1, \infty).$$

The domain of $\vec{r}$ is the intersection of the three domains, namely

$$(-1, 1) \cup (1, \infty)$$

and $\vec{r}$ is continuous on its domain because f, g, h all are.

2 Parametrizing Space Curves

Space Curves

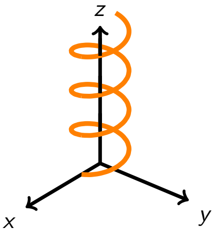
If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, then the curve $\mathcal{C}$ in $\mathbb{R}^3$ with parametric equations
$x = f(t)$, $y = g(t)$, $z = h(t)$ is called the **space curve** parameterized by $\vec{r}$.

Example 3: Describe the space curves:

(I) $\vec{r}(t) = \langle t,\ 1+t,\ 1-t \rangle$: $\mathcal{C}$ is a line

(II) $\vec{r}(t) = \langle \cos(t),\ \sin(t),\ t \rangle$ for $t \geq 0$: $\mathcal{C}$ is a helix

(III) $\vec{r}(t) = \langle t\cos(t),\ t,\ t\sin(t) \rangle$ for $t \geq 0$: $\mathcal{C}$ is a “conical spiral”



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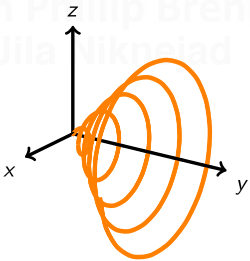

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Space Curves and Parametrizations

Important Note: A single space curve can be parameterized by infinitely many different vector functions! (We've already seen this with lines.)

For example, $\langle \cos(t),\ \sin(t),\ t \rangle$, $\langle \cos(3t),\ \sin(3t),\ 3t \rangle$, $\langle \cos(t^3),\ -\sin(t^3),\ -t^3 \rangle$ all parametrize the same helix.

When studying curves, we often want to distinguish between properties that are **extrinsic** (those that depend on the parametrization) and those that are **intrinsic** (which depend only on the curve itself).

(The length and curviness of the road are intrinsic properties; how fast you are going on it is extrinsic.)

Parametrizing Intersections of Surfaces

Example 4: Find a vector function $\vec{r}(t)$ that parameterizes the curve of intersection of the cylinder $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and the plane $x + y + z = 1$.

<u>Solution:</u> Start by parametrizing the cylinder as

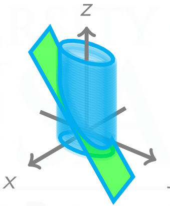
$$x = a\cos(t), \quad y = b\sin(t), \qquad 0 \le t \le 2\pi.$$

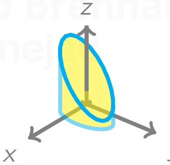
Then use the equation of the plane to find z:

$$z \;=\; 1 - x - y \;=\; 1 - a\cos(t) - b\sin(t).$$

Final answer:

$\vec{r}(t) = \langle a\cos(t), b\sin(t), 1 - a\cos(t) - b\sin(t)\rangle$.
This is not the only parametrization!





Note: The intrinsic equations $x^2/a^2 + y^2/b^2 = 1$ and $x + y + z = 1$ can be recovered from $\vec{r}(t)$ by eliminating the parameter t.

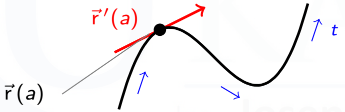
3 Calculus of Vector-Valued Functions

Derivatives of Vector-Valued Functions (Section 13.2)

Given a vector function $\vec{r}(t)$ in $\mathbb{R}^2$ or $\mathbb{R}^3$, the **derivative** of $\vec{r}(t)$ is

$$\vec{r}\,'(t) = \frac{d\vec{r}}{dt} = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

- If $\vec{r}(t)$ is the position of an object at time t, then $\vec{r}\,'(t)$ is its velocity.
- Provided that $\vec{r}\,'(t) \neq \vec{0}$, it is **tangent** to the curve parametrized by $\vec{r}$.



$(x, y) = \vec{r}(t)$

Therefore, the tangent line to the curve of $\vec{r}(t)$ at $t = a$ can be parametrized by the vector function

$$\vec{L}(t) = \vec{r}(a) + t\,\vec{r}\,'(a).$$

Differentiability and Basic Differentiation Rules

Derivatives are computed **component-wise**. If $\vec{r}(t) = \langle f(t),\ g(t) \rangle$ then

$$\vec{r}'(t) = \langle f'(t),\ g'(t) \rangle$$

so $\vec{r}$ is differentiable at t if both f and g are. (The same is true for vector functions in $\mathbb{R}^3$, or for that matter $\mathbb{R}^n$.)

Differentiation rules of single-variable calculus work for vector functions:

$$\frac{d}{dt}\left(\vec{r}(t) + \vec{s}(t)\right) = \vec{r}'(t) + \vec{s}'(t) \qquad \text{(sum rule)}$$

$$\frac{d}{dt}\left(c\vec{r}(t)\right) = c\vec{r}'(t) \qquad \text{(constant multiple rule)}$$

$$\frac{d}{dt}\left(\vec{r}(p(t))\right) = \vec{r}'\big(p(t)\big)\,p'(t) \qquad \text{(Chain Rule)}$$

where $\vec{r}, \vec{s}$ are vector-valued functions, c is any scalar, and p is any scalar-valued function.

Product and Quotient Rules for Vector Functions

There are three **product rules** for vector-valued functions:

Scalar Product Rule: $\dfrac{d}{dt}\left(f(t)\,\vec{\mathrm{r}}(t)\right) = f(t)\,\vec{\mathrm{r}}\,'(t) + f'(t)\,\vec{\mathrm{r}}(t)$

Dot Product Rule: $\dfrac{d}{dt}\left(\vec{\mathrm{r}}(t)\cdot\vec{\mathrm{s}}(t)\right) = \vec{\mathrm{r}}\,'(t)\cdot\vec{\mathrm{s}}(t) + \vec{\mathrm{r}}(t)\cdot\vec{\mathrm{s}}\,'(t)$

Cross Product Rule: $\dfrac{d}{dt}\left(\vec{\mathrm{r}}(t)\times\vec{\mathrm{s}}(t)\right) = \vec{\mathrm{r}}\,'(t)\times\vec{\mathrm{s}}(t) + \vec{\mathrm{r}}(t)\times\vec{\mathrm{s}}\,'(t)$

for any vector functions $\vec{\mathrm{r}}(t)$ and $\vec{\mathrm{s}}(t)$ and any scalar function $f(t)$.

There is one **quotient rule** for the quotient of a vector function by a scalar function.

$$\frac{d}{dt}\left(\frac{\vec{\mathrm{r}}(t)}{f(t)}\right) = \frac{f(t)\,\vec{\mathrm{r}}\,'(t) - f'(t)\,\vec{\mathrm{r}}(t)}{f(t)^2}$$

(Reminder: You can't divide by a vector function!)

Example 1 and A Nifty Fact About Vector Functions

Example 1: Let $f(t) = e^{2t}$ and $\vec{r}(t) = \langle t, t^2, t^3 \rangle$.

(a) $\dfrac{d}{dt}(\vec{r}(t)) = \langle 1, 2t, 3t^2 \rangle$

(b)
$$\begin{aligned} \frac{d}{dt}(f(t)\vec{r}(t)) &= \underbrace{2e^{2t}}_{f'(t)}\underbrace{\langle t, t^2, t^3 \rangle}_{\vec{r}(t)} + \underbrace{e^{2t}}_{f(t)}\underbrace{\langle 1, 2t, 3t^2 \rangle}_{\vec{r}'(t)} \\ &= e^{2t}\langle 2t+1, 2t(t+1), t^2(2t+3) \rangle \end{aligned}$$

(c) $\dfrac{d}{dt}(\vec{r}(f(t))) = \underbrace{2e^{2t}}_{f'(t)}\underbrace{\langle 1, 2e^{2t}, 3e^{4t} \rangle}_{\vec{r}'(f(t))}$

Example 2: $\vec{r}'(t) \perp \vec{r}(t)$ for all t if and only if $\|\vec{r}(t)\| = c$.

Solution: $\dfrac{d}{dt}\left(\|\vec{r}(t)\|^2\right) = \dfrac{d}{dt}(\vec{r}(t) \cdot \vec{r}(t)) = 2\vec{r}'(t) \cdot \vec{r}(t)$.

Thus $\vec{r}'(t) \perp \vec{r}(t)$ if and only if $\vec{r}'(t) \cdot \vec{r}(t) = 0$ if and only if $\dfrac{d}{dt}\left(\|\vec{r}(t)\|^2\right)$ is zero if and only if $\|\vec{r}(t)\|$ is a constant.

An example of such curve in $\mathbb{R}^2$ is a circle and in $\mathbb{R}^3$ is any curve on an sphere: Link.

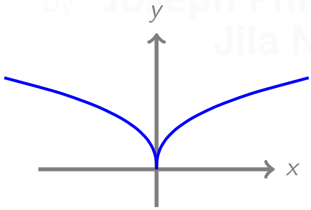
Smooth Curves

- The vector $\vec{r}'(t)$ is tangent to the curve $\mathcal{C}$ parametrized by $\vec{r}(t)$
- The tangent line to $\mathcal{C}$ at $t = a$ can be parametrized as

$$\vec{L}(t) = \vec{r}(a) + t\,\vec{r}'(a)$$

Accordingly, we say that $\mathcal{C}$ is **smooth** on an interval I if $\vec{r}'(t)$ is continuous and nonzero on I (except possibly at the endpoints of I).

Example 3: Let $\vec{r}(t) = \langle t^5, t^2 \rangle$, so that $\vec{r}'(t) = \langle 5t^4, 2t \rangle$. This curve is **not** smooth at $t = 0$, because $\vec{r}'(0) = \vec{0}$.



The Unit Tangent Vector

If a smooth curve $\mathcal{C}$ is parametrized by $\vec{r}(t)$, then the vector function

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}.$$

is called the **unit tangent vector**.

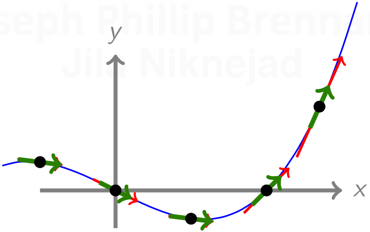
- Note that the curve must be smooth (i.e., $\vec{r}\,'(t) \neq \vec{0}$) for $\vec{T}(t)$ to exist.
- $-\vec{T}(t)$ is also a unit tangent vector.

Example 4:

$\vec{r}(t) = \langle t, t^3 - t \rangle$

$\vec{r}\,'(t) = \langle 1, 3t^2 - 1 \rangle$

$$\vec{T}(t) = \frac{\langle 1, 3t^2 - 1 \rangle}{\sqrt{9t^4 - 6t^2 + 2}}$$



Integrals of Vector Functions (Optional)

Like differentiation, integration of a vector function is defined **component-wise**:

- The **indefinite integral** of $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ is

$$\int \vec{r}(t)\,dt = \left\langle \int f(t)\,dt,\ \int g(t)\,dt,\ \int h(t)\,dt \right\rangle$$

(The integration constants ("$+C$") for each component need not be the same.)

- The **definite integral** of $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ is

$$\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt,\ \int_a^b g(t)\,dt,\ \int_a^b h(t)\,dt \right\rangle$$

Determining Position from Velocity (Optional)

Common application: Given the velocity $\vec{r}'(t)$ of a particle at time t and the position $\vec{r}(0)$ at time $t = 0$, determine the position function $\vec{r}(t)$.

Example 5: Given $\vec{r}'(t) = \langle t, \cos(t), \sin(t) \rangle$, find $\vec{r}(t)$ if $\vec{r}(0) = \langle 0, 0, 0 \rangle$.

Solution: The indefinite integral of $\vec{r}'(t)$ is

$$\vec{r}(t) = \int \vec{r}'(t)\,dt = \left\langle \frac{t^2}{2} + C_1,\ \sin(t) + C_2,\ -\cos(t) + C_3 \right\rangle$$

Now solve for C_1, C_2, C_3 using initial position:

$$\vec{r}(0) = \langle 0, 0, 0 \rangle = \langle C_1,\ C_2,\ -1 + C_3 \rangle \quad \Longrightarrow \quad C_1 = 0,\ C_2 = 0,\ C_3 = 1.$$

Therefore,

$$\vec{r}(t) = \left\langle \frac{t^2}{2},\ \sin(t),\ 1 - \cos(t) \right\rangle$$